Corrigendum
Edge-choosability in line-perfect multigraphs

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Abstract

A multigraph is *line-perfect* if its line graph is perfect. In [1] we claimed that if every edge e of a line-perfect multigraph G is given a list containing at least as many colors as there are edges in a largest edge-clique containing e, then G can be edge-colored from its lists. This note corrects a mistake in our proof.

Keywords: Edge-choosability; List chromatic index; Chromatic index; Edge coloring; List-coloring conjecture; Perfect graph; Perfect line graph; Line-perfect multigraph

We are indebted to Mark Ellingham for informing us of a mistake in our paper [1], which was found by his student Dana Gaston.

Let $G = (V, E)$ be a (finite and loopless) multigraph. An *edge-clique* is a set of mutually adjacent edges, which necessarily consists either of edges incident to some vertex or of edges in (the submultigraph induced by) a clique of three vertices. If $e \in E$, let $\omega'_G(e)$ denote the size of the largest edge-clique containing e. If $v \in V$, define the function $\omega'_{G,v} : E \to \mathbb{N}$ by

$$\omega'_{G,v}(e) := \begin{cases} d_G(v) & \text{if } e \text{ is incident with } v, \\ \omega'_G(e) & \text{otherwise,} \end{cases} \tag{1}$$

where d_G denotes degree in G. We say that G is *edge-lec-choosable* (*lec* standing for the *local edge-clique* number) if, for each $v \in V$, whenever every edge e is given a list of at least $\omega'_{G,v}(e)$ colors, then the edges of G can be properly colored from these lists.

Theorem 4.1 of [1] is an essential step towards the proof of the result stated in the Abstract. It asserts that if G is *of type* $K^*_{1,1,p}$, meaning that its underlying simple graph is of the form $K_{1,1,p}$, then G is edge-lec-choosable. The proof we gave was to color arbitrarily from their lists all the edges between the vertices of the two singleton sets, say x and y, and then to delete these edges from the multigraph and to delete their colors from all other lists to form a bipartite multigraph B. We then claimed that each edge e of B has a list of at least $\omega'_{B,v}(e)$ colors, so that the coloring can be completed by an earlier result from [1], which says that every bipartite multigraph is edge-lec-choosable.

The proof just described is correct if v is x or y, but it fails if v is any other vertex, since an edge of B incident with v may have a list with fewer than $d_B(v)$ $(= \omega'_{B,v}(e))$ colors. The purpose of this corrigendum is to give a proof for this case, which we do by an *ad hoc* argument below.

Let the vertex-set of G be $V = \{x, y, z_1, \dots, z_p\}$, where x and y are the vertices of the singleton sets. Let X_i and Y_i be the sets of edges between x and z_i and between y and z_i respectively, and (in a slight change from the notation in [1]) let Z be the set of edges between x and y. Let $T_i := X_i \cup Y_i \cup Z$ $(i = 1, \dots, p)$. If X is a set of edges, we say that a color is *present on* X if it belongs to the list of at least one edge in X; let $L(X)$ denote the set of colors that are present on X.

We restate and prove the original theorem below.

Theorem 4.1. *Every multigraph whose underlying simple graph is of the form $K_{1,1,p}$ is* *edge-lec-choosable.*

Proof. We will use the notation given above. The proof in [1] is correct if v is x or y. (It follows that the proof is valid for $p = 1$, and so we may assume $p \geqslant 2$.) To complete the proof, we show that edge-lec-choosability holds for any other vertex v, say $v = z_p$. In fact, we will prove a marginally stronger result, namely, that if G is a (not necessarily induced) submultigraph of a multigraph of type $K^*_{1,1,p}$, and every edge e of G is given a list $L(e)$ of colors such that conditions (i)–(iii) below hold, then G can be properly edge-colored from these lists. (Condition (iii) is somewhat weaker than the condition required for edge-lec-choosability of z_p.)

(i) If $1 \leqslant i \leqslant p-1$ then $|L(e)| \geqslant \max\{d(x), |T_i|\}$ for all $e \in X_i$ and $|L(e)| \geqslant \max\{d(y), |T_i|\}$ for all $e \in Y_i$;

(ii) if $e \in X_p \cup Y_p$ then $|L(e)| \geqslant |X_p \cup Y_p| = d(z_p)$;

(iii) if $e \in Z$ then $|L(e)| \geqslant |X_p \cup Y_p \cup Z| = |T_p|$.

Let us assume that G is a counterexample with as few edges as possible. We proceed by a sequence of observations.

(a) $X_p \neq \emptyset$ and $Y_p \neq \emptyset$. For, suppose (say) $X_p = \emptyset$. First color the edges of Y_p from their lists, then color the edges of Z, which can be done by (ii) and (iii). Now the uncolored edges form a bipartite multigraph G', and $|L'(e)| \geqslant \omega'_{G',y}(e)$ for each edge e of G', where $L'(e)$ is the set of colors from $L(e)$ that have not been used on edges adjacent to e. Thus the edges of G' can be colored from their lists by the result of [1] that every bipartite multigraph is edge-lec-choosable. But then we have colored all edges of G from their lists, which contradicts the choice of G as a counterexample to the theorem.

(b) There exist $i, j \in \{1, \ldots, p-1\}$ such that $|T_i| \geqslant d(x)$ and $|T_j| \geqslant d(y)$. For, suppose (say) $|T_i| < d(x)$ for all such i. Color an arbitrary edge $e \in X_p$ with an arbitrary color c from its list. All conditions remain satisfied.

[From now on, when we say that we color an edge with a color c, we assume that we then immediately delete e from G and delete c from the list of every edge adjacent to e that contains c in its list. Let G' be the multigraph obtained by doing this simultaneously for every edge we have colored. If we say that "all conditions remain satisfied", we mean that G' satisfies the hypotheses of the theorem and so can be colored from its lists. This means that G itself can be colored from its lists, which is the required contradiction.]

(c) The i and j mentioned in (b) are unique and equal. For, if we can choose $i \neq j$, then

$$|X_i| + |Y_i| + |Z| = |T_i| \geqslant d(x) \geqslant |X_i| + |X_j| + |X_p| + |Z|$$

and

$$|X_j| + |Y_j| + |Z| = |T_j| \geqslant d(y) \geqslant |Y_i| + |Y_j| + |Y_p| + |Z|,$$

which implies that $|Y_i| \geq |Y_i| + |X_p| + |Y_p|$, contradicting (a). This proves both uniqueness and equality. Assume $i = j = 1$.

(d) $L(X_p) \subseteq L(X_1) \setminus L(Y_1)$ and $L(Y_p) \subseteq L(Y_1) \setminus L(X_1)$. For, if there is a color c in (say) $L(X_p) \cap L(Y_1)$, then use c to color one edge in X_p and one in Y_1; and if there is a color c' in (say) $L(X_p) \setminus L(X_1)$, then use c' to color an edge in X_p. All conditions remain satisfied.

(e) $L(Z) \cap L(X_p) \neq \emptyset$ and $L(Z) \cap L(Y_p) \neq \emptyset$. For, suppose (say) $L(Z) \cap L(X_p) = \emptyset$. By (a) and (d) we can choose colors $c \in L(X_p)$ and $c' \in L(Y_p)$ and use them to color edges $e \in X_1$ and $e' \in Y_p$ respectively. This causes each of $d(x)$, $d(y)$, $d(z_p)$ and $|T_1|$ to decrease by 1. All conditions remain satisfied, since no edge loses more than one color from its list. (Note that, by (d), $L(X_p) \cap L(Y_p) = \emptyset$, $c \notin L(Y_1) \cup L(Z)$, and $c' \notin L(X_p)$.)

(f) $|Z| \geqslant 2$. For, by (e), $Z \neq \emptyset$ and if $|Z| = 1$ then we can color the unique edge in Z with a color $c \in L(X_p)$ and (by (a)) color an edge of Y_p with a color $c' \in L(Y_p)$. Then each of $d(x)$, $d(z_p)$ and $|T_1|$ decreases by 1 and $d(y)$ decreases by 2. The only edges that might lose more than one color from their lists are those in Y_j $(2 \leqslant j \leqslant p-1)$ since $c \notin L(Y_1) \cup L(Y_p)$ and $c' \notin L(X_p)$ by (d), and the colored edge of Y_p is not adjacent to X_j $(1 \leqslant j \leqslant p-1)$. Thus all conditions remain satisfied.

(g) $L(Z) \subseteq L(X_p) \cup L(Y_p)$. For, suppose there is a color $c \in L(Z)$ that is not present on X_p or Y_p. Use c to color an edge $e \in Z$. All conditions remain satisfied.

By (e), (f) and (g), we can choose distinct edges $e_0, e_0' \in Z$ such that $L(e_0)$ contains a color $c_0 \in L(X_p)$ and $L(e_0')$ contains a color $c_0' \in L(Y_p)$ (in which case, by (d), $c_0 \neq c_0'$). Choose edges $e_1 \in X_p$ and $e_1' \in Y_p$. Since $|L(X_p)| \geqslant d(z_p) \geqslant 2$ and similarly $|L(Y_p)| \geqslant 2$, there exist colors $c_1 \in L(e_1) \setminus \{c_0\}$ and $c_1' \in L(e_1') \setminus \{c_0'\}$. Color e_0, e_0', e_1, e_1' with colors c_0, c_0', c_1, c_1' respectively. Then each of $d(x)$ and $d(y)$ decreases by 3, each of $d(z_p)$ and $|T_i|$ $(1 \leqslant i \leqslant p-1)$ decreases by 2, and $|T_p|$ decreases by 4. For each uncolored edge e, if $e \in Z$ then $|L(e)|$ decreases by at most 4; if $e \in X_1 \cup Y_1 \cup X_p \cup Y_p$ then $|L(e)|$ decreases by at most 2 (by (d)); and if $e \in X_i \cup Y_i$ $(2 \leqslant i \leqslant p-1)$ then $|L(e)|$ decreases by at most 3. All conditions remain satisfied, and the theorem is proved. $\square$

References

[1] D. Peterson and D. R. Woodall, Edge-choosability in line-perfect multigraphs, *Discrete Math.* **202** (1999), 191–199.