Edge-choosability in line-perfect multigraphs

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Abstract

A multigraph is *line-perfect* if its line graph is perfect. We prove that if every edge e of a line-perfect multigraph G is given a list containing at least as many colors as there are edges in a largest edge-clique containing e, then G can be edge-colored from its lists. This leads to several characterizations of line-perfect multigraphs in terms of edge-choosability properties. It also proves that these multigraphs satisfy the *list-coloring conjecture*, which states that if every edge of G is given a list of $\chi'(G)$ colors (where χ' denotes the chromatic index) then G can be edge-colored from its lists. Since bipartite multigraphs are line-perfect, this generalizes Galvin's result that the conjecture holds for bipartite multigraphs.

Keywords: edge-choosability, list chromatic index, chromatic index, edge coloring, edge colouring, list-coloring conjecture, list-colouring conjecture, perfect graph, perfect line graph, line-perfect multigraph.

1. Introduction

The *Dinitz Conjecture* was that if each edge of the complete bipartite graph $K_{n,n}$ is assigned a list of n colors, then there is a proper edge coloring in which each edge is assigned a color from its list. This conjecture, and more, was proved by Galvin [4]. Here we generalize Galvin's result to the class of *line-perfect* multigraphs, which includes $K_{n,n}$.

All multigraphs in this paper are finite and loopless. A multigraph $G = (V, E)$ has vertex-set $V = V(G)$, edge-set $E = E(G)$, maximum degree $\varDelta = \varDelta(G)$, chromatic number $\chi = \chi(G)$, chromatic index (edge-chromatic number) $\chi' = \chi'(G)$, clique number (order of a largest clique) $\omega = \omega(G)$ and edge-clique number (size of a largest edge-clique) $\omega' = \omega'(G)$, where a *clique* (resp. *edge-clique*) is a set of mutually adjacent vertices (resp. edges). Note that χ and ω are the same as for the underlying simple graph of G, and if $L(G)$ denotes the line graph of G then $\chi'(G) = \chi(L(G))$ and $\omega'(G) = \omega(L(G))$.

A multigraph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. A multigraph G is *line-perfect* if its line graph $L(G)$ is perfect.

If $(S(v) : v \in V)$ is a family of sets, called *lists*, then we say that G is S*-choosable* or, loosely, that G *can be colored from its lists*, if it is possible to choose an element $c(v) \in S(v)$ for each v such that $c(v) \neq c(w)$ whenever v and w are adjacent. We write $\mathbb{N}_0$ for the set of nonnegative integers. Given a function $f : V \to \mathbb{N}_0$, G is f*-choosable* if G is S-choosable whenever $|S(v)| \geqslant f(v)$ for each $v \in V$, and G is k*-choosable* if it is f-choosable when $f(v) = k$ for each v. The *choice number* or *list chromatic number* $\mathrm{ch} = \mathrm{ch}(G)$ is the smallest k such that G is k-choosable. *Edge-f-choosability* is defined similarly for a function $f : E \to \mathbb{N}_0$, and leads in the same way to the definition of the *edge choice number* or *list chromatic index* $\mathrm{ch}' = \mathrm{ch}'(G)$, which equals $\mathrm{ch}(L(G))$.

Clearly $\omega \leqslant \chi \leqslant \mathrm{ch}$ and $\omega' \leqslant \chi' \leqslant \mathrm{ch}'$ always. Vizing [8] and Erdős, Rubin and Taylor [3] both proved that the difference between ch and χ can be arbitrarily large. But ch' and χ' are equal in every multigraph for which both values are known. The *List-Coloring Conjecture* (*LCC*) is that this always holds: for every multigraph G, $\mathrm{ch}'(G) = \chi'(G)$; see Alon [1] for a history and discussion of the LCC. Observe that the Dinitz Conjecture mentioned in the opening paragraph is a special case of the LCC, in which $G = K_{n,n}$. Galvin [4] proved that the LCC holds for all bipartite multigraphs, which settled the Dinitz Conjecture. We shall prove here that the LCC holds for all line-perfect multigraphs:

Theorem 1.1. *If G is a line-perfect multigraph then* $\mathrm{ch}'(G) = \chi'(G) = \omega'(G)$.

Since bipartite multigraphs are line-perfect (see Theorem 2.1 below), this result generalizes Galvin's theorem. In fact we shall prove a somewhat stronger result. If $e \in E$, let $\omega'_G(e)$ denote the size of the largest edge-clique containing e. Borodin, Kostochka and Woodall [2] strengthened Galvin's theorem by proving that if G is bipartite and each edge $e = uw$ of G is given a list of at least $\max\{d(u), d(w)\}$ colors (where $d(v)$ is the degree of vertex v), then G can be edge-colored from its lists. This result does not extend in this form to all line-perfect graphs, K_3 being an obvious counterexample. However, when G is bipartite, $\max\{d(u), d(w)\}$ can be rewritten as $\omega'_G(e)$, and in this form the result does extend. Clearly $\omega'_G(e) \leqslant \omega' \leqslant \chi' \leqslant \mathrm{ch}'$ for each edge e, and so Theorem 1.1 follows from the following result, which is our main theorem.

Theorem 1.2. *If G is a line-perfect multigraph then G is edge-ω'_G-choosable.*

We shall prove Theorem 1.2 in Sections 2–4. (In fact, we shall prove a marginally stronger result (Theorem 2.3).) In Section 5 we conclude the paper with several characterizations of line-perfect multigraphs in terms of edge-choosability properties.

2. The outer level of the proof

Our starting-point for the proof of Theorem 1.2 is the following theorem, which is easily extracted from the characterization by Maffray ([6], Theorem 2). Here by K_4^* or $K_{1,1,p}^*$ we mean a multigraph whose underlying simple graph is K_4 or $K_{1,1,p}$ $(p \geq 1)$, respectively.

Theorem 2.1. *A multigraph is line-perfect if and only if each of its blocks is bipartite, or a K_4^*, or a $K_{1,1,p}^*$.*

If v is a vertex of a multigraph $G = (V, E)$, define the function $\omega'_{G,v} : E \to \mathbb{N}_0$ by

$$\omega'_{G,v}(e) := \begin{cases} d_G(v) & \text{if } e \text{ is incident with } v, \\ \omega'_G(e) & \text{otherwise,} \end{cases} \tag{1}$$

where d_G denotes degree in G. The following lemma is fundamental.

Lemma 2.2. *Suppose that G is formed from disjoint multigraphs G_1 and G_2 by identifying vertices $w_1 \in G_1$ and $w_2 \in G_2$ into a new vertex w. Suppose that G_1 is edge-$\omega'_{G_1,v}$-choosable and G_2 is edge-ω'_{G_2,w_2}-choosable, where v is any vertex of G_1 (possibly $v = w_1$). Then G is edge-$\omega'_{G,v}$-choosable.*

Proof. Suppose each edge e of G is given a list of at least $\omega'_{G,v}(e)$ colors. Since $\omega'_{G,v}(e) \geqslant \omega'_{G_1,v}(e)$ for each $e \in E(G_1)$, we can edge-color G_1 from these lists. Now a total of $d_{G_1}(w_1)$ colors are used on edges of G_1 at w. Let us remove these colors from the lists on all edges of G_2 at w. If e is such an edge, then the number of colors remaining in its list is at least

$$\omega'_{G,v}(e) - d_{G_1}(w_1) \geqslant d_G(w) - d_{G_1}(w_1) = d_{G_2}(w_2) = \omega'_{G_2,w_2}(e)$$

(regardless of whether $v = w_1$ or not); and each edge e of $G_2 - w_2$ still has a list of at least $\omega'_{G,v}(e) \geqslant \omega'_{G_2,w_2}(e)$ colors. Hence we can complete the coloring, since G_2 is edge-ω'_{G_2,w_2}-choosable. $\square$

We shall say that a multigraph $G = (V, E)$ is *edge-lec-choosable* if it is edge-$\omega'_{G,v}$-choosable for each vertex $v \in V$. (Here *lec* stands for the *local edge-clique* number.) Since $\omega'_{G,v}(e) \leqslant \omega'_G(e)$ for each edge e, the following result obviously implies Theorem 1.2.

Theorem 2.3. *Every line-perfect multigraph is edge-lec-choosable.*

Proof. In the next two sections we shall prove the edge-lec-choosability of every bipartite multigraph, every K_4^*, and every $K_{1,1,p}^*$; that is, every block of a line-perfect multigraph is edge-lec-choosable. Assuming these results, we now prove Theorem 2.3 by induction over blocks.

Let $G = (V, E)$ be a line-perfect multigraph, and let $v \in V$. We must prove that G is edge-$\omega'_{G,v}$-choosable. There is no loss of generality in supposing that G is connected. If G has only one block, then we may assume it is edge-lec-choosable by the previous paragraph. So suppose that G has more than one block. Then it has at least two endblocks (where an *endblock* is a block containing exactly one cutvertex). Let G_2 be an endblock of G not containing v except possibly as its cutvertex w, and let G_1 be the union of all blocks other than G_2. Then $v \in G_1$, and G_1 has one block fewer than G, and so we may suppose by induction that G_1 is edge-lec-choosable, hence edge-$\omega'_{G_1,v}$-choosable. Likewise G_2 is edge-lec-choosable, hence edge-ω'_{G_2,w_2}-choosable, where w_2 is the vertex of G_2 corresponding to w in G. It follows from Lemma 2.2 that G is edge-$\omega'_{G,v}$-choosable. Since this holds for all $v \in V$, it follows that G is edge-lec-choosable, and Theorem 2.3 is proved. $\square$

To complete the inner level of the proof of Theorem 2.3, and hence of Theorem 1.2, we need to prove that every block of a line-perfect multigraph is edge-lec-choosable. This is done in Sections 3 and 4.

3. Bipartite multigraphs

Throughout this section, $G = (V, E)$ will be a bipartite multigraph with partite sets U, W, so that $V = U \cup W$. Let $c : E \to \mathbb{Z}$ be a (proper) edge-coloring of G, to be chosen carefully later. If $e, e' \in E$ we write $e \to e'$ if e, e' are adjacent at a vertex $u \in U$ and $c(e) > c(e')$, or at a vertex $w \in W$ and $c(e) < c(e')$. (It follows that if e, e' are parallel edges then $e \to e'$ and $e' \to e$.) Let $d_c^+(e)$ denote the number of edges e' such that $e \to e'$. The following result is implicit in Galvin's paper [4], and it is stated explicitly (and given a proof from first principles, extracted from [4] and independent of previous references) as Corollary 1.1 of [2].

Theorem 3.1. *If $f(e) > d_c^+(e)$ for each edge e of G, then G is edge-f-choosable.*

Theorem 3.1 immediately implies Galvin's theorem that $\mathrm{ch}'(G) = \chi'(G)$, since if $c : E \to \mathbb{Z}$ is any (proper) edge-coloring of G with $\chi'(G)$ colors then evidently $d_c^+(e) < \chi'(G)$ for each edge e. However, we can get stronger consequences from Theorem 3.1 if we choose c more carefully. For example, if $v \in W$ and we ensure that the edges incident with v get the lowest $d(v)$ colors, then it is easy to see that $d_c^+(e) < d(v)$ for each of these edges. Borodin, Kostochka and Woodall [2] restricted the coloring in a different way in order to prove their result that every bipartite multigraph is edge-ω'_G-choosable. Here we combine both of these ideas in order to extend the result of [2] from edge-ω'_G-choosability to edge-lec-choosability. We use the following result from [2], whose short proof we include for completeness. If M is a matching (a set of nonadjacent edges), then $V(M)$ denotes the set of endvertices of edges in M; and if X is a set of vertices, then $N(X)$ is the set of neighbors of vertices in X. The notation $e = uw$ implies $u \in U$, $w \in W$.

Lemma 3.2. *If $|U| \leqslant |W|$ and W contains no isolated vertices, then G contains a nonempty matching M such that whenever $e = uw \in E$ and $w \in V(M)$, then $u \in V(M)$.*

Proof. Clearly $|N(W)| \leqslant |U| \leqslant |W|$. Let X be a minimal nonempty subset of W such that $|N(X)| \leqslant |X|$. Then $|N(X)| = |X|$ and there is a matching M such that $V(M) = X \cup N(X)$. (If $|X| = 1$, this holds because W contains no isolated vertices. If $|X| \geqslant 2$, it holds by the König–Hall theorem and since $|N(Y)| > |Y|$ whenever $\emptyset \subsetneqq Y \subsetneqq X$.) Clearly M has the required property. $\square$

We are now in a position to prove the main result for bipartite multigraphs, whose proof closely follows the proof of Theorem 3 in [2].

Theorem 3.3. *Every bipartite multigraph is edge-lec-choosable.*

Proof. Let $G = (V, E)$ be a bipartite multigraph where $V = U \cup W$ as before, and let v be an arbitrary vertex of G. We must prove that G is edge-$\omega'_{G,v}$-choosable. In view of Theorem

3.1, it suffices to construct a coloring $c : E \to \mathbb{Z}$ such that

$$d_c^+(e) < \omega'_{G,v}(e) \tag{2}$$

for each edge e. We do this by induction on $|E|$, noting that any coloring c will do if $\Delta(G) = 1$, when $d_c^+(e) = 0 < \omega'_{G,v}(e) = 1$ for each edge e. So suppose $\Delta(G) \geqslant 2$. W.l.o.g. suppose $v \in W$. We may assume that G has no isolated vertices. Note from (1) that if $e = uw$ then

$$\omega'_{G,v}(e) := \begin{cases} d(v) & \text{if } w = v, \\ \max\{d(u), d(w)\} & \text{otherwise,} \end{cases} \tag{3}$$

where $d = d_G$ denotes degree in G.

If $|U| < |W|$ then $|U| \leqslant |W \setminus \{v\}|$. Applying Lemma 3.2 to the graph $G - v$, we deduce that $G - v$ contains a nonempty matching M such that whenever $e = uw \in E$ and $w \in V(M)$, then $u \in V(M)$. By the induction hypothesis, $G \setminus M$ has a coloring $c' : E \setminus M \to \mathbb{Z}$ such that

$$d_{c'}^+(e) < \omega'_{G \setminus M,v}(e) \leqslant \omega'_{G,v}(e) \tag{4}$$

for each edge e. Let $c(e) := c'(e)$ if $e \in E \setminus M$, and if $e \in M$ let $c(e)$ be greater than any color in $c'(E \setminus M)$. If $e = uw \in M$ then clearly $d_c^+(e) < d(u) \leqslant \omega'_{G,v}(e)$ by (3), since $w \neq v$. If $e = uw \notin M$ then (2) follows immediately from (4) if $d_c^+(e) \leqslant d_{c'}^+(e)$. But if $e = uw \notin M$ and $d_c^+(e) > d_{c'}^+(e)$, then $d_c^+(e) = d_{c'}^+(e) + 1$ and $w \in V(M)$; this implies $w \neq v$ and $u \in V(M)$, so that $\omega'_{G,v}(e) = \omega'_{G \setminus M,v}(e) + 1$ by (3). Now (2) follows from (4).

If $|U| \geqslant |W|$ then Lemma 3.2 implies the existence of a nonempty matching M such that whenever $e = uw \in E$ and $u \in V(M)$, then $w \in V(M)$. Define c' and c as before, except that if $e \in M$ we let $c(e)$ now be *less than* any color in $c'(E \setminus M)$. As before, it is easy to see that (2) holds: If $e = uw \in M$ then clearly $d_c^+(e) < d(w) \leqslant \omega'_{G,v}(e)$ by (3), regardless of whether $w = v$ or not. And if $e = uw \notin M$ and $d_c^+(e) > d_{c'}^+(e)$, then $d_c^+(e) = d_{c'}^+(e) + 1$ and $u \in V(M)$; this implies $w \in V(M)$, so that $\omega'_{G,v}(e) = \omega'_{G\setminus M,v}(e) + 1$ by (3), again regardless of whether or not $w = v$. So (2) holds in all cases, and the proof of Theorem 3.3 is complete. $\square$

4. Nonbipartite blocks

In order to complete the proof of Theorem 2.3 it remains to prove that the remaining types of blocks mentioned in Theorem 2.1 are edge-lec-choosable. We do this in the following two theorems.

Theorem 4.1. *Every multigraph of type $K^*_{1,1,p}$ is edge-lec-choosable.*

Proof. Let $G = (V, E)$ be such a multigraph and let v be an arbitrary vertex of G. We must prove that G is edge-$\omega'_{G,v}$-choosable. So let $(S(e) : e \in E)$ be a family of lists with $|S(e)| \geqslant \omega'_{G,v}(e)$ for each edge $e \in E$.

Let x, y be the vertices in the two singleton partite sets, and let X be the set of edges between x and y. Then all edges in X belong to every maximal edge-clique in G, and so $\omega'_{G\setminus X,v}(e) = \omega'_{G,v}(e) - |X|$ for each edge $e \in G \setminus X$. It is easy to color the edges in X from their lists; do so, and let the set of colors used be C. Now

$$|S(e) \setminus C| \geqslant |S(e)| - |X| \geqslant \omega'_{G,v}(e) - |X| = \omega'_{G\setminus X,v}(e)$$

for each edge $e \in G \setminus X$, and since $G \setminus X$ is bipartite, the coloring can be completed by Theorem 3.3. $\square$

To show edge-lec-choosability for K_4^*, it is more convenient to prove the property for a larger class of multigraphs.

Theorem 4.2. *Every multigraph with (exactly) four vertices is edge-lec-choosable.*

Proof. Let $G = (V, E)$ be such a multigraph and let v be an arbitrary vertex of G. We must prove that G is edge-$\omega'_{G,v}$-choosable. We do this by induction on $|E|$, noting that it follows from Theorem 3.3 if $|E| \leq 2$, when G is bipartite. So suppose $|E| \geq 3$, and let $(S(e) : e \in E)$ be a family of lists with $|S(e)| \geqslant \omega'_{G,v}(e)$ for each edge $e \in E$.

Suppose first that G contains two nonadjacent edges e_1, e_2 whose lists have a color in common. Give that color to e_1, e_2 and delete it from the lists of all other edges. Let $H := G \setminus \{e_1, e_2\}$. Since every maximal edge-clique in G contains exactly one of e_1, e_2, it follows that $\omega'_{H,v}(e) = \omega'_{G,v}(e) - 1$ for every edge e of H. Thus H satisfies the induction hypothesis, and we may assume inductively that we can edge-color it from its lists. This gives the required edge-coloring of G.

So we may now suppose that any pair of nonadjacent edges have disjoint sets of colors. In this case we prove that we can give each edge of G a different color by using Hall's theorem to prove that the family of lists has a system of distinct representatives. To do this, it suffices to prove that for every subset $X \subseteq E$, $|S(X)| \geqslant |X|$, where $S(X) := \bigcup_{e \in X} S(e)$. Suppose first that all edges of X are incident with v. Then, for any $e \in X$, $|S(X)| \geqslant |S(e)| \geqslant \omega'_{G,v}(e) = d(v) \geqslant |X|$. Secondly, suppose that the edges of X are mutually adjacent but are not all incident with v. Then, for any $e \in X$ that is not incident with v, $|S(X)| \geqslant |S(e)| \geqslant \omega'_{G,v}(e) = \omega'_G(e) \geqslant |X|$. Finally, if not all edges of X are mutually adjacent, then there must be nonadjacent edges $e_1, e_2 \in X$ such that e_1 is incident with v

and e_2 is not, and then

$$|S(X)| \geqslant |S(e_1)| + |S(e_2)| \geqslant d(v) + |E(G - v)| = |E| \geqslant |X|.$$

Thus $|S(X)| \geqslant |X|$ in all cases, and the family of lists has a system of distinct representatives as required. $\square$

5. Characterizations

We conclude the paper by using Theorem 1.2 to obtain several characterizations of line-perfect multigraphs in terms of edge-choosability properties. It will be useful to recall the following well-known result of Lovász [5].

Theorem 5.1. *If a vertex v in a perfect graph is replaced by a complete graph, all of whose vertices are adjacent to all former neighbors of v, then the resulting graph is perfect.*

Repeated application of Theorem 5.1 to the line graph of a simple graph gives easily:

Corollary 5.2. *A multigraph is line-perfect if and only if its underlying simple graph is.*

The following simple result will also be useful.

Theorem 5.3. *Every submultigraph of a line-perfect multigraph is line-perfect.*

Proof. If G is a (line-perfect) multigraph then removing edges (and, optionally, isolated vertices) from G corresponds to removing vertices from $L(G)$. Since every induced subgraph of a perfect multigraph is also perfect, the result follows. $\square$

Now let $G = (V, E)$ be a multigraph and let $f, g : E \to \mathbb{N}_0$ be two functions. We say that G is *edge-$(f : g)$-choosable* if, for every family $(S(e) : e \in E)$ of lists on edges such that $|S(e)| \geqslant f(e)$ for each $e \in E$, we can find subsets $(T(e) : e \in E)$ with $T(e) \subseteq S(e)$ and $|T(e)| = g(e)$ for each $e \in E$, such that $T(e) \cap T(e') = \emptyset$ whenever e, e' are adjacent. (Here we may replace f or g by an integer denoting the constant function with that value.) Let $\mathcal{C}$ denote the set of all edge-cliques in G. Given $g : E \to \mathbb{N}_0$, let $g(C) := \sum_{e \in C} g(e)$ for each C in $\mathcal{C}$, and define $\widehat{g}, \widehat{g}_{\mathrm{L}} : E \to \mathbb{N}_0$ by

$$\widehat{g}(e) := \max\{g(C) : C \in \mathcal{C}\} \quad \text{and} \quad \widehat{g}_{\mathrm{L}}(e) := \max\{g(C) : e \in C \in \mathcal{C}\}.$$

(So $\widehat{g}$ is a constant function.) Then Theorem 1.2 has the following consequence.

Theorem 5.4. *Let $G = (V, E)$ be a multigraph. Then the following statements are equivalent.*

(a) *G is line-perfect.*

(b) *For every submultigraph H of G, $\mathrm{ch}'(H) = \omega'(H)$.*

(c) *Every submultigraph H of G is edge-ω'_H-choosable.*

(d) *For every function $g : E \to \mathbb{N}_0$, G is edge-$(\widehat{g} : g)$-choosable*

(e) *For every function $g : E \to \mathbb{N}_0$, G is edge-$(\widehat{g}_{\mathrm{L}} : g)$-choosable*

Proof. If G is line-perfect then so is every submultigraph H of G, and so (a) $\Rightarrow$ (c) by Theorem 1.2. Since $\omega'_H(e) \leqslant \omega'(H) \leqslant \chi'(H) \leqslant \mathrm{ch}'(H)$ for each edge e of H, (c) $\Rightarrow$ (b) $\Rightarrow$ (a); thus (a)–(c) are equivalent.

To show (d) $\Rightarrow$ (b), suppose $H \subseteq G$ and define $g : E(G) \to \{0, 1\}$ by $g(e) := 1$ if $e \in H$, 0 if $e \notin H$. Then for each edge $e \in H$, $\widehat{g}(e) = \omega'(H)$, as needed. (If $e \notin H$ then, since $g(e) = 0$, (d) requires no coloring for e from its list.)

Since $\widehat{g}_{\mathrm{L}}(e) \leqslant \widehat{g}(e)$ for each edge e of G, (e) $\Rightarrow$ (d). It remains to prove only that (a) $\Rightarrow$ (e).

So suppose that G is line-perfect, and let $g : E \to \mathbb{N}_0$ and $(S(e) : e \in E)$ be given such that $|S(e)| \geqslant \widehat{g}_{\mathrm{L}}(e)$ for each $e \in E$. Delete each edge e for which $g(e) = 0$, and if $g(e) > 0$ then replace e by a set $X(e)$ of $g(e)$ parallel edges, each of which is given the same list $S(e)$. Call the new multigraph F. Then $\widehat{g}_{\mathrm{L}}(e) = \omega'_F(f)$ for each $e \in E$ and $f \in X(e)$. Moreover, by Theorems 5.1 and 5.3, F is line-perfect. So Theorem 1.2 implies that F is edge-ω'_F-choosable. Therefore we can edge-color F from its lists. If the color on each edge f of F is called $c(f)$, and we now give each edge e of G the set of colors $\{c(f) : f \in X(e)\}$, then we have given e a set of $|X(e)| = g(e)$ colors from $S(e)$, as required for (e). This shows that (a) $\Rightarrow$ (e) and completes the proof of Theorem 5.4. $\square$

Galvin [4] proved that a bipartite multigraph with maximum degree $\varDelta$ is edge-$(k\varDelta : k)$-choosable for every positive integer k; and we can generalize this as follows.

Corollary 5.5. *Let G be a line-perfect multigraph. Then G is edge-$(k\omega' : k)$-choosable and edge-$(k\omega'_G : k)$-choosable for every positive integer k.*

Proof. Apply (d) and (e) of Theorem 5.4 with $g(e) = k$ for every edge e. $\square$

Erdős, Rubin and Taylor [3] asked whether, for $a, b, k \in \mathbb{N}$, every graph that is $(a : b)$-choosable is necessarily $(ka : kb)$-choosable. (Indeed, it seems possible that every graph that is $(f : g)$-choosable is also $(kf : kg)$-choosable.) For *perfect line graphs* (line graphs of line-perfect multigraphs) the answer to their question is yes.

Corollary 5.6. *Let G be a line-perfect multigraph that is edge-$(a : b)$-choosable for some $a, b \in \mathbb{N}$. Then G is edge-$(ka : kb)$-choosable for every positive integer k.*

Proof. We first observe that a line-perfect multigraph G is edge-$(c:d)$-choosable if and only if $c \geqslant d\omega'$. The "if" part follows from Corollary 5.5 and the "only if" part follows by assigning every edge in a maximum edge-clique the same list. We now have the series of implications: G is edge-$(a:b)$-choosable $\Rightarrow a \geqslant b\omega' \Rightarrow ka \geqslant kb\omega' \Rightarrow G$ is edge-$(ka:kb)$-choosable. $\square$

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